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A study of the action from kinematical integral geometry point of view

M. A. DEL OLMO AND M. SANTANDER
Departamento de Física Teórica. Facultad de Ciencias
Universidad de Valladolid
47011 - VALLADOLID, SPAIN

Abstract. *We develop here an interpretation of the classical nonrelativistic and relativistic action for a point particle as related to geometric measures of sets of straight lines (inertial motions) associated in a natural way to closed timelike circuits in space-time. This allows a point of view for the action common to classical and relativistic mechanics. Furthermore the results are not restricted to the free case and also holds for particles in some potentials (homogeneous field and the harmonic oscillator).*

1. INTRODUCTION

Action is the single most important quantity in physics [1] and it is worth to explore any path that could rise new points of view about it. That is the aim of this paper.

The relativistic action for a free particle has a well-known geometrical meaning, as it is proportional to the length of (proper time along) the particle worldline. But in the non-relativistic case, the length of a worldline (the lapse of universal time), is path independent. Action is therefore introduced in classical mechanics as a path dependent quantity, without any known geometrical meaning. The relationship between both actions is described in geometric terms by a «timelike» contraction from Minkowski space-time: in the proximity of a timelike straight line, the proper time along a wordline with fixed

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end-points has a path-independent dominant term (the «universal lapse of time»), and the next small term (in an expansion in powers of $1/c$) is path dependent, and equals, up to a factor, the classical action.

Our purpose here is to discuss in some detail the extension of known results of (Euclidean) Integral Geometry [2] to the context of space-time geometry (Galilean or Minkowskian). Unexpectedly, these questions turn out to be related with the action for a point particle in both cases. In spite of its elementary nature, and although Integral Geometry results are known for Klein spaces [3], it seems that this connection has not been explicitly pointed out previously. Some of the results have been announced without proofs [4, 5], which are given here.

We take as starting point some well-known relations in (Euclidean) integral geometry, giving the length of a closed circuit as an integral over the set of straight lines, each line taken with a weight equal to the number of times it intersects the circuit. This idea cannot be immediately translated to kinematical geometries, (e.g. Galilean or Minkowskian), because it leads to divergent integrals. We show, however, that a proper accounting of orientation of lines, which is an intrinsic property in kinematics, allows a new definition of the number of oriented intersections, and with this definition divergences do not appear. To be more specific, if a timelike circuit in the $(1+1)$ Galilean (Minkowski) space-time is obtained as the union of two future timelike paths from A to B, the integral over the set of future, timelike straight lines, each of them taken with a weight equal to the oriented number of intersections with the circuit, has a finite value which equals (up to a factor) the difference of the classical (relativistic) actions of a free particle along the two trajectories. This is shown in Sect. 2. Section 3 relates these results to the Gauss-Bonnet theorem. In Section 4 we consider a particle in an external field. The results here show that for potentials given by polynomials up to second order (i.e. «free fall» in a homogeneous field and the harmonic oscillator), the difference of actions along two timelike paths with common endpoints - now of course, with the potential term equals the integral of the number of oriented intersections over the set of real motions. In Section 5 we extend these results to the $(1+2)$ and $(1+3)$ cases.

The main moral of this paper, both for the free and for some interacting cases, is that the quantity with a well-defined meaning in the sense of integral geometry is not the action for an open path, but (in a formal way), the action along a closed timelike path. All this is very satisfactory, as this is precisely the content of the classical concept «action» which turns out to be essential in Quantum Mechanics as the relative phase of two paths in Feynmann's integration [6]. It is at least remarkable that a geometrical analysis points to the quantity which we know to be the really relevant one and that this quantity has a geometric origin related with a circuit as is also the case for the quantum Berry's phase [7]. The interest of such quantities associated with loops is well known in Quantum Mechanics and in gauge theories, and it is perhaps surprising that a similar kind of idea lies hidden in Classical Mechanics.

2. INTEGRAL GEOMETRY FOR (1 + 1) KINEMATICAL GROUPS (FREE CASE)

2.1. Integral relations for the euclidean case.

A complete exposition can be found in [2], and here we will only recall some relations which will be helpful for our purposes. Let M^q be a q- dimensional compact differentiable manifold in n- dimensional euclidean space ($n \geq q$). If ℓ_r denotes a r- plane in E_n, then $\ell_r \cap M^q$ is a manifold whose dimension is, in general, $\leq r + q - n$. In the case $r + q - n = 0$, the intersection generically reduces to a point. Let us denote by $N(\ell_r \cap M^q)$ the number of points of intersection of M^q with the generic r- plane ℓ_r and by $\mathrm{d}\ell_r$ the density determined in the set of all r- planes by the condition of being invariant under the n- dimensional euclidean group. For this case the following integral formula holds:

$$\int_{\ell_r \cap M^q \neq \emptyset} N(\ell_r \cap M^q)\, \mathrm{d}\ell_r = \frac{O_n \cdots\cdots O_{n-r+1}}{O_r \cdots O_1} \sigma_q(M^q) \,, \tag{2.1}$$

where O_i is the measure of the i- dimensional sphere in the euclidean space, and $\sigma_q(M^q)$ is the measure of the manifold M^q ; O_i is given by:

$$O_i = \frac{2\pi^{(i+1)/2}}{\Gamma((i+1)/2)} \,.$$

For $n = 2$ and $q = r = 1$, any (unoriented) straight line in the euclidean plane is parametrized by the angle θ between the segment perpendicular from the origin O and a fixed straight line through the origin, and by the length p of that segment. The density $\mathrm{d}\ell_1$ (hereafter denoted $\mathrm{d}\ell$), is $\mathrm{d}\ell = \mathrm{d}p \wedge \mathrm{d}\theta$. Let Γ be an interval of length L_Γ on a straight line m. Every other straight line will meet Γ at once or none at all. The integral in (2.1) equals to twice the length of the segment:

$$\int_{\ell \cap \Gamma \neq \emptyset} \mathrm{d}p \wedge \mathrm{d}\theta = 2L_\Gamma \,. \tag{2.2}$$

The most widely known consequence of (2.2) is the so-called Cauchy-Crofton formula: Let Γ be a piecewise differentiable closed curve. If $N_\Gamma(\ell)$ denotes the number of intersections of the line ℓ with Γ, we have:

$$\int N_\Gamma(\ell)\, \mathrm{d}\ell = 2L_\Gamma \,. \tag{2.3}$$

A formally identical formula also holds for spherical and hyperbolic geometries, where the corresponding manifold of points (the «plane») is some homogeneous space of the groups $SO(3)$ or $SO(2,1)$. Even for any two-dimensional Riemannian space, similar formulas can be derived [2]. But as far as we know, the question of whether there is any physically meaningful integral relations for the kinematical geometries (Galilei or Minkowski) has not been discussed. This is what we explore in the next section.

Other formulas which will be useful in Sec. 5 correspond to the case where M is a curve of length L, denoted M^1, and ℓ_r are $(n-1)$- dimensional planes.

(2.4) $$\overset{n=3,\,r=2}{\quad} \int_{\ell_2\cap M^1\neq\emptyset} N(\ell_2\cap M^1)\,\mathrm{d}\ell_2 = \pi L$$

(2.5) $$\overset{n=4,\,r=3}{\quad} \int_{\ell_3\cap M^1\neq\emptyset} N(\ell_3\cap M^1)\,\mathrm{d}\ell_3 = \frac{4}{3}\pi L \quad .$$

2.2. The Galilei and Minkowski cases.

In the Galilean plane, $\mathcal{G}$, we introduce coordinates (t,x), in such a way that the action of $G(1,1)$ on $\mathcal{G}$ is given by:

(2.6) $$(b,a,v) : \begin{bmatrix} t \\ x \end{bmatrix} \to \begin{bmatrix} t+b \\ x+vt+a \end{bmatrix}$$

There are two kinds of straight lines in that geometry: ordinary timelike lines, $x = kt+s$, $k,s\in\mathbf{R}$, and special (null or spacelike) lines, $t=t_o$. The group $G(1,1)$ acts transitively on each of these kinds of lines, the action of $G(1,1)$ on the two-dimensional set $\mathcal{G}^*$ of timelike lines (dual of the Galilean plane, or «cogalilean plane») is:

(2.7) $$(b,a,v) : \begin{bmatrix} k \\ s \end{bmatrix} \to \begin{bmatrix} k+v \\ s+(-b)k+(a-bv) \end{bmatrix}$$

The only (up to a factor) two-form on $\mathcal{G}^*$ invariant under the action of $G(1,1)$ is $\mathrm{d}l = \mathrm{d}k\wedge\mathrm{d}s$.

In the Minkowski case we introduce coordinates (t,x) in such a way that the action of $M(1,1)$ on $\mathcal{M}$ is given by:

(2.8) $$(b,a,\chi) : \begin{bmatrix} t \\ x \end{bmatrix} \to \begin{bmatrix} tCh\chi+xSh\chi+b \\ tSh\chi+xCh\chi+a \end{bmatrix}$$

There are in $\mathcal{M}$ three kinds of straight lines: timelike lines, $x=kt+s$, $-1<k<1$, $s\in\mathbf{R}$, whose set is denoted $\mathcal{M}^*$; null lines, $x=\varepsilon t+s$, $\varepsilon\in\{1,-1\}$, $s\in\mathbf{R}$; and space-like lines. The Poincaré group $M(1,1)$ acts transitively on each of these three kinds of lines. On $\mathcal{M}^*$, the line $tSh\zeta + xCh\zeta - p = 0$ can be denoted (ζ,p). $M(1,1)$ acts on $\mathcal{M}^*$ as:

(2.9) $$(b,a,\chi) : \begin{bmatrix} \zeta \\ p \end{bmatrix} \to \begin{bmatrix} \zeta+\chi \\ p+bSh(\zeta+\chi)+aCh(\zeta+\chi) \end{bmatrix}$$

and $d\zeta \wedge dp$ is an invariant two-form on $\mathcal{M}^*$.

In both cases the attempt of finding a naive generalization of formula (2.2) fails, because of the non-compact nature of the subgroup of «rotations» (galilean or Lorentz boosts).

There is, however, an important difference between Galilean and Minkowskian geometries on one side, and Euclidean on the other one: the (time) orientability properties of the sets of lines. Each line can be given, in all cases, two orientations, and while Euclidean group still acts transitively on the set of oriented lines this is not the case for Galilei and Poincaré groups. There, the set of oriented timelike lines splits into two orbits, called future and past lines. This is also reflected on the topological structure of the set of all (timelike) unoriented lines, which is itself orientable for Galilean and Minkowskian geometries, but not for the Euclidean one. That distinction on future and past straight lines extend of course in an invariant way to all timelike curves.

Let us now consider, either in the Galilei or the Minkowski $(1+1)$ space-time, two piecewise differentiable open future timelike curves Γ_1 and Γ_2, with common endpoints. In an evident sense the pair (Γ_1, Γ_2) can be considered as a closed, timelike, piecewise differentiable curve, with a future timelike part Γ_1 and a past timelike part, denoted $-\Gamma_2$.

As the distinction between future and past curves is of geometric nature, invariant under the group $G(1,1)$ or $M(1,1)$, we can assign an orientation to the intersection of curves in a consistent way. The intersection of two timelike curves is said positively oriented if the two curves are both future (past) at the intersection point, and negatively oriented if one of them is future and the other past. To each intersection of two timelike curves we assign an intersection number, $n = 1$ or $n = -1$ whether the intersection is either positively or negatively oriented.

In the Euclidean plane there is no possibility of making an intrinsic election of orientation of lines, and hence the idea of an «oriented intersection number» is not intrinsic. Anyway, we can always split any closed curve Γ by means of two points A and B on Γ, and take the two arcs Γ_1, Γ_2 with lengths L_1, L_2 and common endpoints A and B, as having opposite orientations. For every straight line, say ℓ we count each intersection with Γ_1 as 1 and each intersection with Γ_2 as -1, and define an «oriented total intersection number», $N_\Gamma(\ell)$ as the sum of all the intersection numbers of the straight line ℓ with Γ (figure 1). From (2.2) it is easy to derive the following integral relation:

$$\int N_\Gamma(\ell)\, d\ell = 2(L_1 - L_2) \tag{2.10}$$

In the Euclidean case this formula involves an arbitrary partition of the closed curve Γ. For Galilei and Minkowski cases, however, the partition in future and past arcs is intrinsic, and hence the number of oriented intersections of any timelike line with Γ is

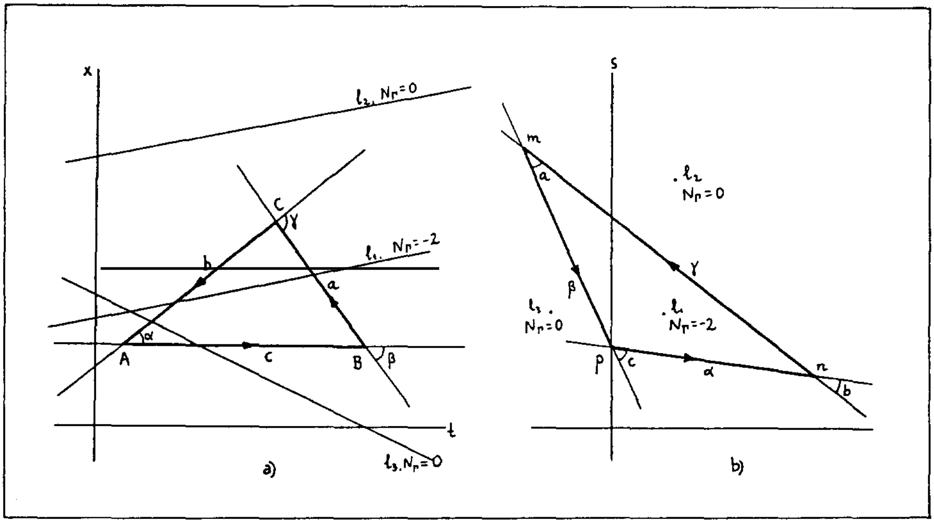

Fig. 1. a) A «pure» triangle in the galilean plane. Lines l_1, l_2, l_3 have the oriented intersection numbers N_Γ as shown. b) Its dual triangle in the cogalilean plane.

well-defined. The interesting point is that the integral over $\mathcal{G}^*$ analogous to the one in the l.h.s. of (2.10) is free of divergences.

Consider any closed timelike curve Γ in the classical $(1+1)$ space-time obtained from two open future timelike curves Γ_1, Γ_2 with common endpoints, A, B. As the arc length on these lines is simply the coordinate t (universal time), there is no restriction if we take the curves Γ_i, $i = 1,2$ to be:

$$t \in [t_A, t_B] \to (t, x_i(t)), \quad \text{with} \begin{cases} x_1(t_A) = x_2(t_A) \\ x_1(t_B) = x_2(t_B) \end{cases} \tag{2.11}$$

THEOREM 1. *Let Γ_i, $i = 1,2$ be the two differentiable future timelike curves in the galilean plane given by (2.11). If $N_\Gamma(\ell)$ denotes the total oriented number of intersections of the future timelike straight line ℓ with $\Gamma \equiv (\Gamma_1, \Gamma_2)$, the following relation holds:*

$$\int N_\Gamma(\ell)\, d\ell = \int_{t_A}^{t_B} \{(\dot{x}_1(t))^2 - (\dot{x}_2(t))^2\}\, dt \tag{2.12}$$

PROOF. In Section 5 we will carry out very similar computations for the case of a particle in an external field, and we shall now prove Theorem 1 following a method which will be applicable there. We implicitly assume that all the conditions needed to perform in a legitimate way the transformations in the integrals (Fubini theorem, interchange of the order of integrations, etc.) are satisfied. The situation is depicted in Figure 2.

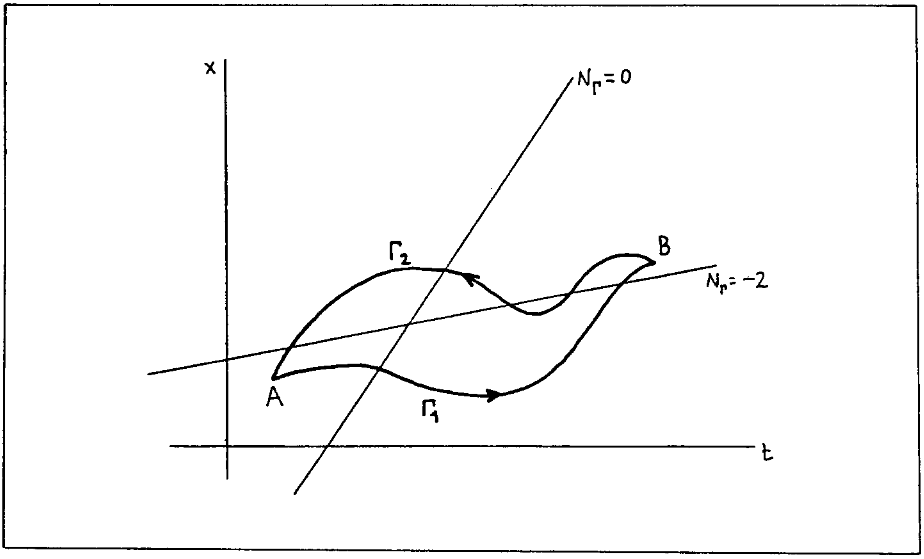

Fig. 2. A closed timelike loop in Galilean plane with some total (oriented) intersection numbers.

We shall start by assuming that coordinates (k, s) have been selected in the set $\mathcal{G}^*$ in such a way that the invariant density is simply $\mathrm{d}\,k \wedge \mathrm{d}\,s$. (This is certainly the case for the parametrization used in (2.7)). The condition of incidence of a point $(t, x_i(t))$ on the curve Γ_i and a generic line ℓ can be considered as the equation of a curve in the set $\mathcal{G}^*$. That curve in the «dual plane» is (at least locally) the graph of some function $s = s_i(k, t, x_i(t))$. When t varies, this curve moves in the dual plane, but the points $(k(t), s(t))$ satisfying the equations $s = s_i$, $\mathrm{d}\,s_i/\mathrm{d}\,t = 0$ remain stationary. In our present case and the ones which we shall discuss in section 5, the equation $\mathrm{d}\,s_i(k, t, x_i(t))/\mathrm{d}\,t = 0$ has a single solution for k-denoted $k_i(t)$ – , and hence a single stationary point $(k_i(t), s_i(k_i, t, x_i(t)))$. The integral over $\mathcal{G}^*$ of the number of intersections of a generic straight line (k, s) with Γ_1 is:

$$H_1 = \int_{t_A}^{t_B} \mathrm{d}\,t \left[\int_{-\infty}^{k_1(t)} \frac{\mathrm{d}\,s_1(t)}{\mathrm{d}\,t} \mathrm{d}\,k - \int_{k_1(t)}^{+\infty} \frac{\mathrm{d}\,s_1(t)}{\mathrm{d}\,t} \mathrm{d}\,k \right] . \tag{2.13}$$

A similar expression holds for the integral of the number of intersections with Γ_2. As these have to be counted with an intersection number (with Γ) $n = -1$, the integral of the total oriented number of intersections with Γ is:

$$\begin{aligned} H = \int_{t_A}^{t_B} \mathrm{d}\,t \Bigg[& \int_{-\infty}^{k_1(t)} \frac{\mathrm{d}\,s_1(t)}{\mathrm{d}\,t} \mathrm{d}\,k - \int_{k_1(t)}^{+\infty} \frac{\mathrm{d}\,s_1(t)}{\mathrm{d}\,t} \mathrm{d}\,k - \\ & - \int_{-\infty}^{k_2(t)} \frac{\mathrm{d}\,s_2(t)}{\mathrm{d}\,t} \mathrm{d}\,k + \int_{k_2(t)}^{+\infty} \frac{\mathrm{d}\,s_2(t)}{\mathrm{d}\,t} \mathrm{d}\,k \Bigg] . \end{aligned} \tag{2.14}$$

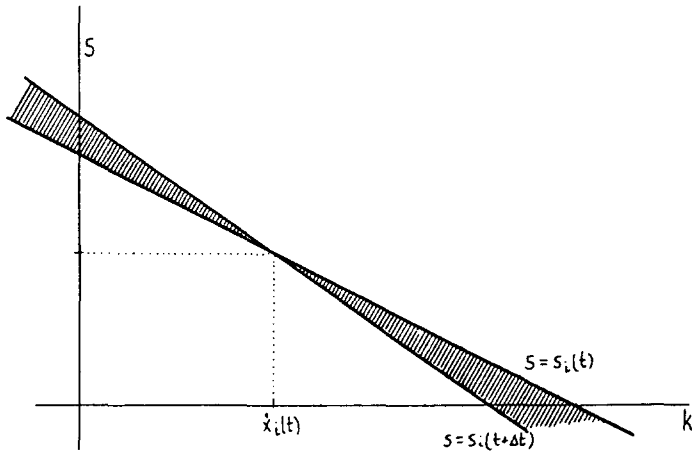

Fig. 3. «Instantaneous» motion of the line which wipes out the effective integration domain in the cogalilean plane.

In the first integral we replace the interval $(-\infty, k_1(t))$ by $(-\infty, k_2(t)) \cup (k_2(t), k_1(t))$, and in the last one $(k_2(t), \infty)$ by $(k_2(t), k_1(t)) \cup (k_1(t), \infty)$. After some reduction we get:

$$(2.15)\qquad \begin{aligned} H = \int_{t_A}^{t_B} \mathrm{d}t \Bigg[&- \int_{k_2(t)}^{-k_1(t)} \frac{\mathrm{d}(s_1(t) - s_2(t))}{\mathrm{d}t} \mathrm{d}k + \\ &+ \int_{k_2(t)}^{k_1(t)} \frac{\mathrm{d}(s_1(t) + s_2(t))}{\mathrm{d}t} \mathrm{d}k \Bigg] . \end{aligned}$$

This is a general result. Returning to the present case, the generic line ℓ, whose equation is $x = kt + s$, passes through the point $(t, x_i(t))$ if and only if $s = x_i(t) - kt$. The function $s_i(t)$ is then $s_i(t) \equiv x_i(t) - kt$. This is the equation of a straight line in the cogalilean plane, whose motion when t varies is a rotation with center $k_i(t) = \dot{x}_i(t)$ given by the unique root of $\mathrm{d}s_i/\mathrm{d}t = 0$ (see Figure 3). By substitution of these values in the general expression (2.15):

$$(2.16)\qquad \begin{aligned} H = \int_{t_A}^{t_B} \mathrm{d}t \Bigg\{ &\int_{-\dot{x}_2(t)}^{\dot{x}_1(t)} (\dot{x}_1(t) - \dot{x}_2(t)) \mathrm{d}k + \\ &\int_{\dot{x}_2(t)}^{\dot{x}_1(t)} (\dot{x}_1(t) + \dot{x}_2(t) - 2k) \mathrm{d}k \Bigg\} \mathrm{d}t . \end{aligned}$$

The integrals in k are now free of divergences. The second integral vanishes, thus the final expression of the integral of the total number of oriented intersections with Γ is:

$$H = \int N_\Gamma(k,s)\,\mathrm{d}k \wedge \mathrm{d}s = \int_{t_A}^{t_B} \{(\dot{x}_1(t))^2 - (\dot{x}_2(t))^2\}\,\mathrm{d}t \tag{2.17}$$

This formula also holds when Γ_1, Γ_2 are only piecewise differenciable arcs.

3. THE ACTION AS A GEOMETRIC QUANTITY AND THE GAUSS-BONNET THEOREM

The simplest example of the Gauss-Bonnet Theorem appears for triangles in spherical and hyperbolic geometries, where the relation

$$KS = (\widehat{A} + \widehat{B} + \widehat{C} - \pi) \tag{3.1}$$

links the (internal) angles and the area of the triangle. For the euclidean case (3.1) reduces to a trivial identity, and S is unrelated to the (zero) angular excess. In terms of the three angles $\alpha = \widehat{A}$, $-\gamma = \widehat{C} - \pi$ and $\beta = \widehat{\beta}$ (which are the angles turned at points A, B, C by the path ACB as one starts from the straight line $p \equiv AB$ at A and goes back to the same line at B), the relation (3.1) is

$$KS = \alpha + \beta - \gamma\,. \tag{3.2}$$

Note that γ is the «exterior» angle at the vertex which is not on p. Under this form, the «triangle» version of Gauss-Bonnet theorem holds also for the geometries of all lines (timelike when applicable) in the euclidean, galilean and minkowskian $(1+1)$ space-times, whose constant curvatures are respectively, positive, zero and negative (in fact (3.2) holds for all two-dimensional Cayley-Klein geometries; see [8,9] for the Gauss-Bonnet theorem in the minkowskian case, where the angles α, β, γ receive a direct physical interpretation as rapidity changes [10], and [11-15] for Cayley-Klein geometries).

We show here the close relationship between the results in the former section and this form of Gauss-Bonnet theorem. Consider a triangle either in the euclidean, galilean or minkowskian plane, in the two last cases of the «pure» type (only future time-like lines allowed, see [11] for the Minkowski case) where AB, AC, CB are (future timelike) segments on the straight lines p, n, m respectively with lengths c, b, a. This triangle is to be looked at as a circuit Γ with AB as Γ_1, and the two segments AC and CB as Γ_2. The three lines p, n, m in space-time correspond to three points p, n, m which determine an oriented triangle, $\Delta(pnm)$ in the dual coeuclidean, co-galilean or co-minkowskian plane. The (oriented) number of intersections of a generic (future) straight line with the closed path Γ is constant on the convex domains into which the three sides of the triangle pnm divides the dual plane. Except for the subset (of zero measure) of points on these three lines, the oriented number of intersections with Γ is either equal to 0 or to -2, depending on whether the point in the dual plane is either outside or inside the triangle $\Delta(pnm)$ (see figures 1a, 1b). For the integral of the total number of oriented intersections of a generic (future, timelike) line with Γ over the set of all these lines we obtain:

$$\int N_\Gamma(k,s)\,\mathrm{d}k\wedge\mathrm{d}s = -2\int_{\Delta(pnm)}\mathrm{d}k\wedge\mathrm{d}s = -2\ \text{Area}\ \Delta(pnm)\ . \tag{3.3}$$

The area $\Delta(pnm)$ is linked through formula (3.2) to the angle excess of the oriented triangle pnm. But the angles α,β,γ of this dual triangle are the lengths of the sides a,b,c of the original triangle, so that we finally obtain

$$K^*\Delta(pnm) = a+b-c\,, \tag{3.4}$$

where K^* can be reduced to the values $1,0,-1$ respectively for the Euclidean, Galilean and Minkowskian cases by adjusting the units of measure of angles and lengths. Henceforth, combining (3.3) and (3.4):

$$\left|\int N_\Gamma(\ell)\,\mathrm{d}\ell\right| = 2\,|\text{Area}\ \Delta(pnm)| = \tag{3.5}$$
$$= \begin{cases} 2(a+b-c) = 2(L_1-L_2), & \text{euclidean case}\\ \int_{t_A}^{t_B}\{(\dot x_1(t))^2-(\dot x_2(t))^2\}\,\mathrm{d}t, & \text{galilean case}\\ -2(a+b-c) = 2(L_2-L_1), & \text{minkowskian case}\end{cases}$$

Notice the natural appearance of the non-relativistic action for a free particle as the galilean counterpart of the euclidean/minkowskian difference between lengths in the same integral formula (3.5).

The extension of the preceding reasoning to more general curves affords another way to proof Theorem 1, through a standard limiting process starting from triangles. Let us first consider the special case where the first path Γ_1 from A to B is along the straight line AB and Γ_2 is along a future time like curve, $\tau \to (\tau, x_2(\tau))$, with $t\in(0,T)$ and $x_2(0)=x_A$, $x_2(T)=x_B$ (see Figures 4a, 4b). There is no loss of generality if we take $x_A=0$, $x_B=0$ as another case can be reduced to this one by some appropriate motion which leaves invariant the measures. For every value of τ (to be considered as an evolution parameter), the time like line tangent to Γ_2 satisfies the equation:

$$x_{2,\tau}(t) = \frac{\mathrm{d}x_2(\tau)}{\mathrm{d}\tau}t + \left\{x_2(\tau) - \tau\frac{\mathrm{d}x_2(\tau)}{\mathrm{d}\tau}\right\} \equiv k_2(\tau)t + s_2(\tau)\,,$$

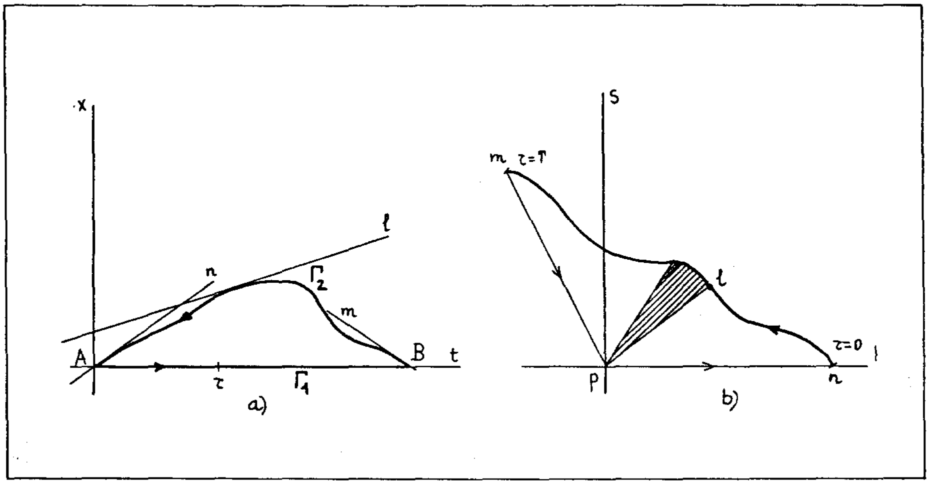

Fig. 4. a) A timelike closed circuit in the Galilean plane. b) Its dual in the cogalilean plane.

so that the evolution from A to B along Γ_2, corresponds to the curve

$$\tau \longrightarrow \begin{bmatrix} \frac{\mathrm{d}x_2(\tau)}{\mathrm{d}\tau} \\ x_2(\tau) - \tau\frac{\mathrm{d}x_2(\tau)}{\mathrm{d}\tau} \end{bmatrix} \tag{3.6}$$

in the cogalilean plane. As previously the integral of the total number of oriented intersections with $\Gamma \equiv (\Gamma_1, \Gamma_2)$ over the set $\mathcal{G}^*$ equals -2 times the area of the domain bounded in the cogalilean plane by the straight line segment pn, the curve (3.6) and the segment mp. Using galilean trigonometry [12] we obtain for this area S^* the expression:

$$|S^*| = -\frac{1}{2}\int_0^T x_2(\tau)\frac{\mathrm{d}^2x_2(\tau)}{\mathrm{d}\tau^2}\,\mathrm{d}\tau\,.$$

Integration by parts and the boundary conditions for $x_2(\tau)$ gives the following result:

$$|S^*| = \frac{1}{2}\int_0^T \left[\frac{\mathrm{d}x_2(\tau)}{\mathrm{d}\tau}\right]^2 \mathrm{d}\tau\,. \tag{3.7}$$

If Γ_1 were not the straight path from A to B but a future timelike curve, the corresponding domain in the cogalilean plane would be bounded by two curves like (3.6) and two straight segments. The immediate extension of our previous discussion gives for the area of the corresponding domain:

$$|S^*| = \frac{1}{2}\int_0^T \left(\left[\frac{\mathrm{d}x_2(\tau)}{\mathrm{d}\tau}\right]^2 - \left[\frac{\mathrm{d}x_1(\tau)}{\mathrm{d}\tau}\right]^2\right)\mathrm{d}\tau \tag{3.8}$$

By substitution of (3.8) into (3.3) we obtain the relation (2.12) in theorem 1.

4. INTEGRAL GEOMETRY FOR PARTICLES IN AN EXTERNAL FIELD

For a free particle, action along a closed path $\Gamma \equiv (\Gamma_1, \Gamma_2)$ has a geometrical meaning embodied in eq. (2.12), where the integral $\int N_\Gamma(\ell)\,\mathrm{d}\,\ell$ is taken over the straight lines in the Galilean plane, i.e., the worldlines of the free motions. Is relation (2.12) an accidental coincidence for a free particle or a particular case of a general law which remains valid also for particles interacting with an external field? The study of this problem is worthy, since the difference of actions is the phase factor giving the contribution of every history of the particle in Feynmann's formulation of Quantum Mechanics.

Most of the results in Euclidean Integral Geometry also hold for any 2-dimensional Riemannian space $\mathcal{R}$ [2], whose set of geodesics $\mathcal{R}^*$ is not in general a G-homogeneous space. Thus, the condition of group invariance for the density of geodesics does not play an essential role. In fact, there is always a natural symplectic structure in the space $\mathcal{R}^*$ of geodesics [16] and only for a Riemannian space of constant curvature can $\mathcal{R}^*$ be also considered as a G-homogeneous symplectic space (G is either $SO(2,1)$, $E(2)$ or $SO(3)$). In these cases, the symplectic structure of $\mathcal{R}^*$ can be readily found using the well known Kostant-Kirillov-Souriau theorem [17], since $\mathcal{R}^*$ is diffeomorphic to a coadjoint orbit of G .

A similar situation is well known in classical mechanics: the phase space has always a natural symplectic structure, which is only invariant under «kinematical» groups in some particular situations. We consider in this section the classical, Galilean $(1+1)$ case. The space of lines which is the space of motions [18] and the phase space are naturally identified. To see it, fix an instant t_0 and let p, x respectively be the momentum and position at $t = t_0$ on the line. Then the Liouville measure $\mathrm{d}\,p \wedge \mathrm{d}\,x$ in phase space is equal - up to a factor - to the Galilei invariant measure $\mathrm{d}\,k \wedge \mathrm{d}\,s$ on the set of lines.

Therefore, it seems natural to study the case of a particle in a given external potential from the point of view of Section 2. We do not discuss here the most general potential $V(x)$, but restrict ourselves to an intermediate situation, corresponding to linear dynamics, where $V(x)$ is a time independent quadratic polynomial in x . For these cases, the space of actual motions with its natural symplectic structure can be identified to a coadjoint orbit of some group G . For $V^{(1)}(x) = -gx$ and $V^{(2)}(x) = (1/2)kx^2$, G is respectively, the extended Galilei and the Newton-Hooke $(1+1)$ group [19,20]. A G-invariant density for the set of actual motions can be found, either directly or through the study of the coadjoint orbits of G . When the space of actual motions is identified to the phase space, this G-invariant density is equal - up to a factor - to the Liouville measure. Then, (sections 4.1 and 4.2; see also [5]), the integral $\int N_\Gamma(\ell)\mathrm{d}\,\ell$ relative to the Liouville measure $\mathrm{d}\,\ell$ is equal to the difference between actions along Γ_1 and Γ_2 *in the potential* $V(x)$:

$$
\begin{aligned}
\int N_\Gamma(\ell)\,\mathrm{d}\ell = 2\int_{t_A}^{t_B} \Big[& \left\{ \frac{1}{2}(\dot{x}_1(t))^2 - V(x_1(t)) \right\} - \\
- & \left\{ \frac{1}{2}(\dot{x}_2(t))^2 - V(x_2(t)) \right\} \Big]\,\mathrm{d}t\,.
\end{aligned}
\tag{4.1}
$$

The presence of the Liouville measure in this formula raises the question about the validity of (4.1) for an arbitrary potential beyond linear dynamics. This is an open problem. However, some numerical simulations for the one-dimensional Kepler problem, which will appear in a forthcoming paper [21], strongly suggests that (4.1) is also valid there.

4.1. Motion in a uniform field.

Let us consider a uniform force field, with a potential $V^{(1)}(x) = -gx$. As the set of «lines» we take the solutions of the equations of motion, namely:

$$
x = s + kt + \frac{1}{2}gt^2 \tag{4.2}
$$

This set is a two-dimensional manifold, which can be parametrized by (k, s). Despite of the fact that for a general potential the Galilei group does not act in the set of actual motions, this potential appears as an exception, and a simple calculation shows that

$$
(b, a, v) : \begin{bmatrix} k \\ s \end{bmatrix} \longrightarrow \begin{bmatrix} k + v - gb \\ s + a - kb - vb + (1/2)gb^2 \end{bmatrix}. \tag{4.3}
$$

In the ordinary interpretation of galilean geometry, the subgroup $\{(b, 0, 0)\}$ in the action (4.3) corresponds to displacements along the cycles $s = s_0 + (1/2g)k^2$ and the subgroup $\{(b, -(1/2)gb^2, gb)\}$ corresponds to the ordinary galilean rotations with center $k = 0$,

$$
(b, -(1/2)gb^2, gb) : \begin{bmatrix} k \\ s \end{bmatrix} \to \begin{bmatrix} k \\ s - kb \end{bmatrix}.
$$

From the results in Section 2 $\mathrm{d}k \wedge \mathrm{d}s$ is the unique (up to a factor) two-form invariant under this action, which is also the Liouville measure in phase space. From the point of view of coadjoint orbits, the extended $(1 + 1)$ Galilei group is a 5-dimensional Lie group [19, 22] whose elements will be written (θ, η, b, a, v); if (m, f, E, p, k) is an element of the dual of its Lie algebra, the coadjoint action is [22]:

$$
\begin{aligned}
m' &= m \\
f' &= f \\
E' &= E + \frac{1}{2} m v^2 - vp - af \\
p' &= p - mv + bf \\
k' &= k + \frac{1}{2} b^2 f + (a - bv) m + bp
\end{aligned}
$$

and has three invariants, m, f, and $p^2 - 2mE - 2fk$. There are five strata, and each orbit in the stratum with $m \neq 0, f \neq 0$ is symplectomorphic to the space of motions in the case under consideration. Introducing canonical coordinates p, $q = k/m$, a simple computation shows that the (unique up to a factor) symplectic two-form associated to that orbit is $\mathrm{d}p \wedge \mathrm{d}q$. The symplectomorphism between that orbit and the space of all motions being given by $k = p/m, s = q$, the two-form $\mathrm{d}k \wedge \mathrm{d}s$ in coordinates k, s in the space of all motions is recovered. Incidentally, note that the orbits in the stratum with $m \neq 0, f = 0$ corresponds to the free particle, and that the expressions for the canonical coordinates in the orbit and for the symplectic form coincide with the former ones and give the measure $\mathrm{d}k \wedge \mathrm{d}s$ for free motions as discussed previously.

We consider now a general closed timelike curve, obtained with two open timelike curves with common endpoints, A, B as in (2.11) and which for the sake of simplicity we also assume to be differentiable. We shall prove equation (4.1) with $V^{(1)}(x) = -gx$, using the pattern of the proof of Theorem 1, and starting from equation (2.15). A point $(t, x_i(t))$ meets the line (k, s) if and only if for s we have $s = x_i(t) - kt - (1/2)gt^2 \equiv s_i(t)$. This is also the equation of a line in the (k, s) plane. The equation $\mathrm{d}s_i/\mathrm{d}t = 0$ has for fixed t the unique solution $k_i = (\mathrm{d}x_i(t)/\mathrm{d}t) - gt$. Inserting these values in (2.15) we obtain:

$$
\begin{aligned}
H = \int_{t_A}^{t_B} \mathrm{d}t \Bigg\{ &\int_{-k_2(t)}^{k_1(t)} (\dot{x}_1(t) - \dot{x}_2(t))\, \mathrm{d}k + \\
+ &\int_{k_2(t)}^{k_1(t)} (\dot{x}_1(t) + \dot{x}_2(t) - 2k - 2gt)\, \mathrm{d}k \Bigg\} .
\end{aligned}
\tag{4.4}
$$

As in the free case the integrals converge. When performing the final integration in t, a further integration by parts in the second integral taking into account $gx_1(t) =$

$-V^{(1)}(x_1(t))$, gives:

$$H = \int N_\Gamma(\ell)\, \mathrm{d}\ell = \int_{t_A}^{t_B} \{(\dot{x}_1(t))^2 - (\dot{x}_2(t))^2 + 2gt(\dot{x}_1(t) - \dot{x}_2(t))\}\, \mathrm{d}t = 2\int_{t_A}^{t_B} \left[\left\{\frac{1}{2}(\dot{x}_1(t))^2 - V^{(1)}(x_1(t))\right\} - \left\{\frac{1}{2}(\dot{x}_2(t))^2 - V^{(1)}(x_2(t))\right\}\right] \mathrm{d}t\,. \tag{4.5}$$

4.2. The harmonic oscillator.

The harmonic oscillator also allows a fully explicit calculation [5] of the integrals of the number of oriented intersections; the invariance group is not a subgroup of the Galilei group, but the Newton-Hooke $(1+1)$ group (isomorphic to the plane euclidean group [20]) which acts in the set of the oscillator actual motions. The details of the identification of the space of actual motions with a coadjoint orbit of this group are as follows: the extended group is a four dimensional Lie group, and denoting by (m, E, p, k) a generic element of the dual of the Lie algebra, the coadjoint action [22] is:

$$\begin{aligned}
m' &= m \\
E' &= E + \frac{1}{2}(\omega^2 a^2 + v^2)m - vp + \omega^2 ak \\
p' &= p\cos(\omega b) - (v\cos(\omega b) + a\omega\sin(\omega b))m - \omega^2 k \sin(\omega b) \\
k' &= k\cos(\omega b) + (a\cos(\omega b) + (v/\omega)\sin(\omega b))m - (p/\omega)\sin(\omega b)\,.
\end{aligned}$$

The invariants are m , $p^2 + \omega^2 k^2 - 2mE$, and there are three strata. The relevant here is the one with $m \neq 0$, and a simple calculation shows that in coordinates p , $q = k/m$ the symplectic two-form is $\mathrm{d}p \wedge \mathrm{d}q$. When translated to the coordinates k, s used in [5] reduces again to $\mathrm{d}k \wedge \mathrm{d}s$ up to a factor. The final result, analogous to (4.5) is

$$\int N_\Gamma(\ell)\mathrm{d}\ell = 2\int_{t_A}^{t_B} \left[\left\{\frac{1}{2}(\dot{x}_1(t))^2 - \frac{1}{2}\omega^2(x_1(t))^2\right\} - \left\{\frac{1}{2}(\dot{x}_2(t))^2 - \frac{1}{2}\omega^2(x_1(t))^2\right\}\right] \mathrm{d}t \tag{4.6}$$

which coincides with (4.1) for the potential $V^{(2)}(x) = (1/2)\omega^2 x^2$.

5. INTEGRAL GEOMETRY FOR $(1+2)$ and $(1+3)$ GALILEI GROUPS

First, let us consider the action of the $(1+2)$ Galilei group in the space-time $\mathcal{G}_2$ with two space dimensions. We choose coordinates (t, x) , in such a way that the action of $G(1,2)$ on $\mathcal{G}_2$ is:

(5.1) $$(b, \mathbf{a}, \mathbf{v}, \varphi) : \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} \to \begin{bmatrix} t + b \\ R_\varphi \mathbf{x} + \mathbf{v}t + \mathbf{a} \end{bmatrix} .$$

There are two kinds of 2-planes here: «space-like» 2-planes, which are the leaves of the Newtonian space-time stratification, $t = t_0$, and «inertial» planes whose equation is:

(5.2) $$\cos\theta \, x^1 + \sin\theta \, x^2 - kt + p = 0 ,$$

where $p \in \mathbf{R}$, $k \in \mathbf{R}$, $\theta \in (-\pi/2, \pi/2]$. Each inertial plane has itself a natural $(1 + 1)$ galilean geometry, and hence contains timelike lines and spacelike lines, and can also be given two different orientations; the $(1 + 2)$ Galilei group acts transitively on each of the subsets of oriented planes. We do not write down the explicit form of this action (which can be done using the coordinates (p, k, θ)), but it is easy to convince oneself that the three-form $\mathrm{d}\,\ell_2 = \mathrm{d}\,p \wedge \mathrm{d}\,k \wedge \mathrm{d}\,\theta$ is the unique (up to a factor) three-form invariant under that action. From this density for the oriented «inertial» 2-planes (which are indeed the generic ones), one can consider the measure of the set of inertial 2-planes which intersect a given timelike segment. This is given by a divergent integral, as in the $(1 + 1)$ case. Nevertheless, if we consider the integral over «inertial» 2-planes of the total number of oriented intersections with a general closed timelike curve the divergence disappears. There is no restriction in taking the arcs Γ_1, Γ_2 to be:

(5.3) $$t \in [t_A, t_B] \to (t, x_i(t)), \text{ with } \begin{cases} x_1(t_A) = x_2(t_A) \\ x_1(t_B) = x_2(t_B) \end{cases}$$

We also assume that the two arcs are differentiable, although the final main result holds for piecewise differentiable curves.

THEOREM 2. *Let* Γ_i, $i = 1,2$ *be the two differentiable future timelike curves (5.3) in the galilean* $(1 + 2)$ *space-time. If* $N_\Gamma(k, s)$ *denotes the total oriented number of intersections of the inertial 2-plane* ℓ_2 *with* $\Gamma \equiv (\Gamma_1, \Gamma_2)$, *the following integral relation holds:*

(5.4) $$\int N_\Gamma(\ell_2)\, \mathrm{d}\,\ell_2 = 2\pi \int_{t_A}^{t_B} \{(1/2)(\dot{x}_1(t))^2 - (1/2)(\dot{x}_2(t))^2\}\, \mathrm{d}\,t .$$

The proof of this result follows the same pattern than in the $(1 + 1)$ case. The condition of intersection of the generic point $(t, x_i(t))$ on the curve $x_i(t)$, and a 2-plane $\ell_2 \equiv (k, p, \theta)$ can be considered as the equation of a surface in the set of all 2-planes which can be given locally as some function $p = p_i(k, \theta, t, x_i(t))$. The explicit expression is:

(5.5) $$p_i(t) = -\cos\theta \, x^1{}_i - \sin\theta \, x^2{}_i + kt .$$

When t varies, the points on this surface with satisfy the equations $p = p_i$, $\mathrm{d}p_i/\mathrm{d}t = 0$ determine a curve in the set of all 2-planes which takes the role of the point $k_i(t)$ in the $(1+1)$ case. The equation of that curve, which is indeed a straight line, is:

$$\cos\theta\, x^1{}_i + \sin\theta\, x^2{}_i - k = 0\,.$$

Let us denote $k_i(\theta, t)$ the function which gives k as function of θ, t along this curve, i.e.,

$$k_i(\theta,t) = \cos\theta\, x^1{}_i + \sin\theta\, x^2{}_i\,. \tag{5.6}$$

The integral of the total number of oriented intersections with each path Γ_i is therefore given by the expression:

$$\begin{aligned} H_i = \int_{t_A}^{t_B} \mathrm{d}t \Bigg[&\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{-\infty}^{k_i(t)} \frac{\mathrm{d}p_i(t)}{\mathrm{d}t}\mathrm{d}k - \\ &- \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{k_i(t)}^{+\infty} \frac{\mathrm{d}p_i(t)}{\mathrm{d}t}\mathrm{d}k \Bigg]\,. \end{aligned} \tag{5.7}$$

The integral H over all inertial 2-planes of the total number of oriented intersections with $\Gamma \equiv (\Gamma_1, \Gamma_2)$ is:

$$\begin{aligned} \int_{t_A}^{t_B} \Bigg\{ &\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{-\infty}^{k_1(t)} \frac{\mathrm{d}p_1(t)}{\mathrm{d}t}\mathrm{d}k - \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{k_1(t)}^{+\infty} \frac{\mathrm{d}p_1(t)}{\mathrm{d}t}\mathrm{d}k - \\ &- \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{-\infty}^{k_2(t)} \frac{\mathrm{d}p_2(t)}{\mathrm{d}t}\mathrm{d}k + \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{k_2(t)}^{+\infty} \frac{\mathrm{d}p_2(t)}{\mathrm{d}t}\mathrm{d}k \Bigg\}\, \mathrm{d}t\,. \end{aligned}$$

With the same procedure than in the $(1+1)$ case, and after some reduction the final expression is:

$$\begin{aligned} H = \int_{t_A}^{t_B} \Bigg\{ &- \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{k_2(t)}^{-k_1(t)} \frac{\mathrm{d}(p_1(t) - p_2(t))}{\mathrm{d}t}\mathrm{d}k + \\ &+ \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{k_2(t)}^{k_1(t)} \frac{\mathrm{d}(p_1(t) + p_2(t))}{\mathrm{d}t}\mathrm{d}k \Bigg\}\, \mathrm{d}t\,, \end{aligned} \tag{5.8}$$

which upon substitution of the expressions (5.5) and (5.6) gives for the integral of the total number of oriented intersections the result announced in Theorem 2:

$$\begin{aligned} H = \int N_\Gamma(\ell_2)\, \mathrm{d}\ell_2 = 2\pi \int_{t_A}^{t_B} &\{(1/2)(\dot{x}_1(t))^2 - \\ &- (1/2)(\dot{x}_2(t))^2\}\, \mathrm{d}t\,. \end{aligned} \tag{5.9}$$

The discussion for the $1+3$ space-time case is quite similar to the above. The action of the group is given in coordinates $(t, \mathbf{x})$ by the equations (5.1) with $\mathbf{x} \in \mathbb{R}^3$, and the rotation $R \in SO(3)$. There are two kinds of 3-planes, «spacelike» ones $(t = t_0)$ and «inertial» 3-planes parametrized by $0 \leq \theta \leq \pi$, $-\pi/2 \leq \varphi \leq \pi/2$, k, $p \in R$, and with equation:

$$\cos\varphi \sin\theta\, x^1 + \sin\varphi \sin\theta\, x^2 + \cos\theta\, x^3 - kt + p = 0\ . \tag{5.10}$$

The invariant measure $d\ell_3$ on the set of all these 3-planes is, up to a factor, given by $d\ell_3 = \sin\theta\, dp \wedge dk \wedge d\theta \wedge d\varphi$. Using formula (2.5) we obtain for the integral of the oriented number of intersections a result quite similar to the one in (5.4). This result can be summarized in the following statement:

THEOREM 3. *Let Γ_i, $i = 1,2$ be the two differentiable future timelike curves in the galilean $(1+3)$ space-time as in (5.3). If $N_\Gamma(k,s)$ denotes the total oriented number of intersections of the inertial 3-plane ℓ_3 with $\Gamma \equiv (\Gamma_1, \Gamma_2)$, the following integral relation holds:*

$$\int N_\Gamma(\ell_3)\, d\ell_3 = (8\pi/3) \int_{t_A}^{t_B} \{(1/2)(\dot{x}_1(t))^2 - (1/2)(\dot{x}_2(t))^2\}\, dt \tag{5.11}$$

The proof of this theorem is an almost verbatim transcription of the preceding case and is left to the reader.

The linear structure of equations (5.4) and (5.11) allows the decomposition of its r.h.s. as a sum of three integrals, each of them with the structure of the r.h.s. of (2.12). Each of these terms corresponds to one of the (two or three) orthogonal projections of the paths Γ_1, Γ_2 on the two-dimensional space-time planes $X_i - T (i = 1,2$ or $1,2,3)$. Thus, we have proved for the $3+1$ case the following statement:

$$(3/4\pi) \int N_\Gamma(\ell_3)\, d\ell_3 = \int N_{\Gamma,(1)}(\ell_1)\, d\ell_1 + \\ + \int N_{\Gamma,(2)}(\ell_1)\, d\ell_1 + \int N_{\Gamma,(3)}(\ell_1)\, d\ell_1\ ,$$

where $N_{\Gamma,(i)}(\ell_1)$ stands for the total number of oriented intersections of the projection of Γ on the plane $X_i - T$ with the generic straight timelike line ℓ_1 on this plane. Each integral in the r.h.s. is performed over this set of lines. Similar expressions holds for the $1+2$ case.

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